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STEADY-STATE ALGORITHMIC ANALYSIS
OF
M/M/c TWO-PRIORITY QUEUES WITH HETEROGENEOUS RATES.

by

10 Douglas R./Miller

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Introduction

This paper presents an algorithm for computing steady-state probabilities and mean queue lengths and delay times for M/M/c queues with two priority classes. Cobham [2] has computed mean delay times for multipriority M/M/c queues with homogeneous service rates. Heyman [3] pointed out the need to compute mean delay times where service rates for different priority classes are unequal. The algorithm presented below computes expectations for a system with two priority classes having heterogeneous rates; in addition, it can compute steady state probabilities for individual states. However it is doubtful whether this algorithm could be modified to a practical algorithm for systems involving more than two priority classes. The preemptive case is simpler than the nonpreemptive case; an algorithm for it can be developed using ideas presented in this paper.

The general approach to this analysis is a computational method developed by Neuts [8]. The simpler problem of single-server two priority Markovian queues was solved by Miller [5]. The approach consists of partitioning the full state space into blocks and discovering special structure of the invariant probability vector in relation to these blocks. It is assumed that the reader is familiar with [5,6,7,8].

In addition to solving the problem posed by Heyman, this algorithmic approach is of interest because it explores a new and complicated application of several ideas from the field of computational probability: quasi birth and death processes, M/G/1 paradigms, and matrix-geometric invariant vectors.

Using single precision arithmetic the algorithm appears to work well for low and moderate utilization factors, but there is some deterioration in the calculations for high utilization. This application demonstrates the need for developing good error analyses for this type of computation. This application may be a good vehicle for developing some such analyses.

State Space and Transition Matrices

Consider an M/M/c two priority nonpreemptive queueing system with arrival rates λ_1 and λ_2 and service rates μ_1 and μ_2. The state space can be described as follows. Let $x_{i,j,k}$ be the state with i first priority customers waiting, j second priority customers waiting, k first priority customers in service and $c-k$ second priority customers in service, $i, j \geq 0$, $0 \leq k \leq c$. From them, define the blocks

$$H_{i,j} = \{x_{i,j,k} | 0 \leq k \leq c\} \quad , \quad i,j \geq 0 \; .$$

Let $x_{m,n}$ be the state with m customers in service of which n are first priority customers and no customers are waiting in queue, $0 \leq n \leq m \leq c$. Then define the blocks

$$H_m = \{x_{m,n} | 0 \leq n \leq m\} \; , \quad 0 \leq m \leq c \; .$$

(Note that $H_c = H_{0,0}$.) Thus, the state space is

$$S = \left(\bigcup_{m=o}^{c-1} H_m \right) \cup \left(\bigcup_{i \geq o} \bigcup_{j \geq o} H_{i,j} \right) \; .$$

Now consider the probabilistic transition rates for the non-preemptive M/M/c two priority system on this state space. The matrices of transition rates between the blocks defined above can be denoted as

$$K_i: \quad H_i \to H_{i+1} \; , \qquad 0 \leq i \leq c-1$$

$$J_i: \quad H_i \to H_{i-1} \; , \qquad 1 \leq i \leq c$$

$$L_1: \quad H_{i,j} \to H_{i+1,j} \; , \qquad i,j \geq 0$$

$$L_2: \quad H_{i,j} \to H_{i,j+1} \; , \qquad i,j \geq 0$$

$$M_1: \quad H_{i,j} \to H_{i-1,j} \; , \qquad i \geq 1, \quad j \geq 0$$

$$M_2: \quad H_{0,j} \to H_{0,j-1} \; , \qquad j \geq 1 \; .$$

All transitions correspond to arrivals and departures thus there are no transitions within individual blocks. Therefore the submatrices of the transition rate matrix corresponding to transitions within a block are all diagonal:

$$-D_i: \quad H_i \longrightarrow H_i \,, \qquad 0 \le i \le c$$

$$-D_c: \quad H_{i,j} \longrightarrow H_{i,j} \,, \qquad i,j \ge 0$$

The transition structure is depicted in Figure 1.

The matrix K_i is $(i+1) \times (i+2)$ dimensional; $(K_i)_{j,k}$ equals λ_1 for $k = j+1$, λ_2 for $k = j$, and 0 otherwise $(1 \le j \le i+1$, $1 \le k \le i+2)$. The matrix J_i is $(i+1) \times i$ dimensional; $(J_i)_{j,k}$ equals $(i-j+1)\mu_2$ for $k = j$, $(j-1)\mu_1$ for $k = j-1$, and 0 otherwise $(1 \le j \le i+1$, $1 \le k \le i)$. The matrices L_1, L_2, M_1, and M_2 are $(c+1) \times (c+1)$ dimensional; L_1 equals $\lambda_1 I$; L_2 equals $\lambda_2 I$; $(M_1)_{j,k}$ equals $(c-j+1)\mu_2$ for $k = j+1$, $(j-1)\mu_1$ for $k = j$, and 0 otherwise $(1 \le j \le c+1$, $1 \le k \le c+1)$; $(M_2)_{j,k}$ equals $(c-j+1)\mu_2$ for $k = j$, $(j-1)\mu_1$ for $k = j-1$, and 0 otherwise $(1 \le j \le c+1$, $1 \le k \le c+1)$. The matrix D_i is $(i+1) \times (i+1)$ dimensional; $(D_i)_{j,k}$ equals $(j-1)\mu_1 + (i-j+1)\mu_2 + \lambda$ for $k = j$ and 0 otherwise $(1 \le j \le i+1$, $1 \le k \le i+1)$.

The blocks H_m, $0 \le m \le c-1$, and $H_{i,j}$, $i,j \ge 0$, can be combined into super blocks:

$$I_{-1} = \bigcup_{i=o}^{c-1} H_i \,, \quad I_i = \bigcup_{j=o}^{\infty} H_{i,j} \,, \quad i \ge 0 \,.$$

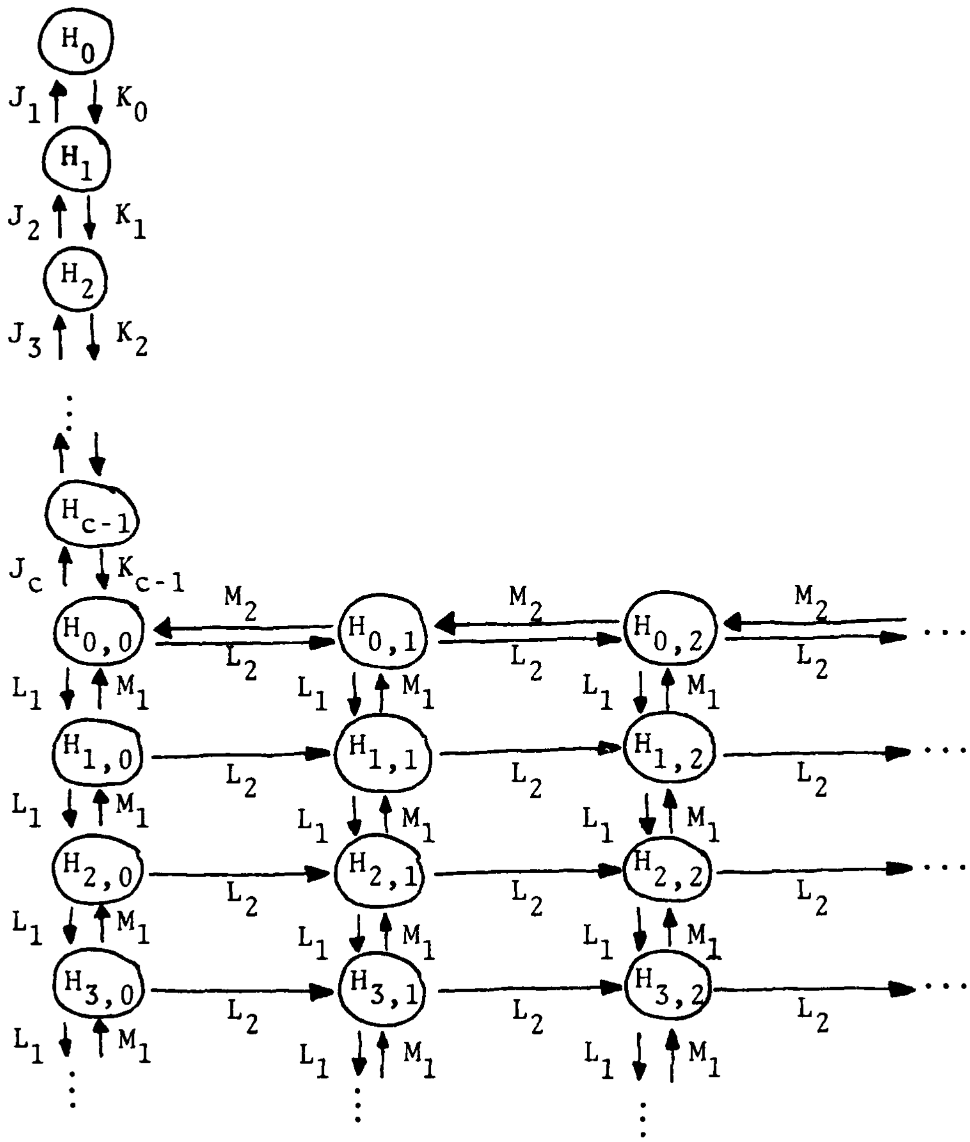

Figure 1. Partitioned state space (blocks) and transition scheme.

The states in I_{-1} are exactly those corresponding to the existence of idle servers. The states in I_i , $i \geq 0$, are those with exactly i first priority customers awaiting service. The matrices of transition rates between and within these blocks can be denoted as

$$
\begin{array}{lll}
B_{-1,-1}: & I_{-1} \longrightarrow I_{-1} & \\
B_{-1,0}: & I_{-1} \longrightarrow I_0 & \\
B_{0,-1}: & I_0 \longrightarrow I_{-1} & \\
B_{0,0}: & I_0 \longrightarrow I_0 & \\
A_0: & I_i \longrightarrow I_{i+1} , & i \geq 0 \\
A_1: & I_i \longrightarrow I_i , & i \geq 1 \\
A_2: & I_i \longrightarrow I_{i-1} , & i \geq 1 .
\end{array}
$$

These transitions are depicted in Figure 2. The above matrices of transition rates can each be partitioned into submatrices corresponding to transitions between the subblocks of I_i , $i \geq -1$. These submatrices were defined earlier. The number of submatrices in one of these matrices varies according to the number of blocks in the corresponding superblocks. $(B_{-1,-1})_{j,k}$ equals K_{j-1} for $k = j+1$, $-D_{j-1}$ for $k = j$, J_{j-1} for $k = j-1$, and 0 otherwise $(1 \leq j \leq c , 1 \leq k \leq c)$. $(B_{-1,0})_{j,k}$ equals K_{c-1} for $(j,k) = (1,c)$ and 0 otherwise $(1 \leq j < \infty , 1 \leq k \leq c)$. $(B_{0,-1})_{j,k}$ equals J_c for $(j,k) = (c,1)$ and 0 otherwise $(1 \leq j \leq c , 1 \leq k < \infty)$. $(B_{0,0})_{j,k}$ equals L_2 for $k = j+1$, $-D_c$ for $k = j$, M_2 for $k = j-1$, and 0 otherwise $(1 \leq j < \infty , 1 \leq k < \infty)$. $(A_0)_{j,k}$ equals L_1 for $k = j$ and 0

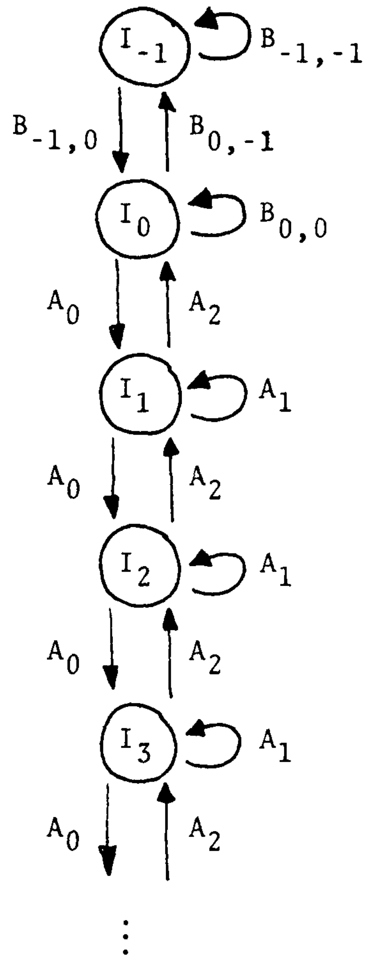

Figure 2. Partitioned state space (superblocks) and transition scheme.

otherwise $(1 \le j < \infty,\ 1 \le k < \infty)$. $(A_1)_{j,k}$ equals L_2 for $k = j+1$, $-D_c$ for $k = j$, and 0 otherwise $(1 \le j < \infty,\ 1 \le k < \infty)$. $(A_2)_{j,k}$ equals M_1 for $k = j$ and 0 otherwise $(1 \le j < \infty,\ 1 \le k < \infty)$.

Using this block structure and notation, the transition rate matrix for the M/M/c two-priority nonpreemptive queueing system is

$$P_S = \begin{pmatrix} B_{-1,-1} & B_{-1,0} & 0 & 0 & 0 & \cdots \\ B_{0,-1} & B_{0,0} & A_0 & 0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \qquad (1)$$

<u>Invariant Measures</u>

The process described above is a quasi birth and death process [4,8]. Consequently its steady-state probability vector is of matrix-geometric form [4,8]:

$$(\underline{\pi}_{-1},\ \underline{\pi}_0,\ \underline{\pi}_0 R,\ \underline{\pi}_0 R^2,\ \underline{\pi}_0 R^3,\ \ldots) .$$

The Matrix-Geometric Rate Matrix

The rate matrix, R , of the quasi birth and death process is the minimal solution of

$$A_0 + RA_1 + R^2A_2 = 0 . \tag{2}$$

Furthermore, from the block structure of the process and the interpretation of the rate matrix [5,7,8], R must have the structure

$$R = \begin{pmatrix} R_0 & R_1 & R_2 & R_3 & R_4 & \cdots \\ 0 & R_0 & R_1 & R_2 & R_3 & \cdots \\ 0 & 0 & R_0 & R_1 & R_2 & \cdots \\ 0 & 0 & 0 & R_0 & R_1 & \cdots \\ 0 & 0 & 0 & 0 & R_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{3}$$

where each submatrix has dimension $(c+1) \times (c+1)$. Substituting (3) into (2) gives a system of equations:

$$L_1 - R_0D_c + R_0^2M_1 = 0 \tag{4}$$

$$R_{i-1}L_2 - R_iD_c + \sum_{j=0}^{i} R_jR_{i-j}M_1 = 0 \tag{5}$$

These can be solved numerically as follows. First consider (4). Let

$$\tilde{L}_1 = D_c^{-1}L_1 \ , \quad \tilde{M}_1 = D_c^{-1}M_1 \ , \quad \tilde{R}_0 = D_c^{-1}R_0D_c \ ;$$

then (4) becomes

$$\tilde{R}_0 = \tilde{L}_1 + \tilde{R}_0^{\,2}\,\tilde{M}_1 \tag{6}$$

which can be solved using the usual iterative approach useful for matrix-geometric rate matrices [8]: Let

$$S_0 = 0 \ , \quad S_1 = \tilde{L}_1 + S_0^{\,2}\,\tilde{M}_1 \ , \ \ldots \ , \ S_{i+1} = \tilde{L}_1 + S_i^{\,2}\,\tilde{M}_1 \ , \ \ldots$$

Then $S_i \uparrow \tilde{R}_0$, termwise. The usual procedure is to continue iterating until the maximum termwise difference between successive iterates is smaller than some ε , e.g. $\varepsilon = 10^{-7}$.

Now consider (5); it is equivalent to

$$\tilde{R}_1 = \tilde{R}_{i-1}\tilde{L}_2 + \sum_{j=1}^{i-1} \tilde{R}_j\tilde{R}_{i-j}\tilde{M}_1 + (\tilde{R}_0\tilde{R}_i + \tilde{R}_i\tilde{R}_0)\,\tilde{M}_1 \ , \ i \geq 1 \tag{7}$$

For $i = 1$, $\tilde{R}_1$ can be found by using the above solution for $\tilde{R}_0$ and then using a similar iterative procedure which will converge monotonically to $\tilde{R}_1$. For $i \geq 1$, continue recursively, using the solutions for $\tilde{R}_j$, $j < i$, from previous steps and using the iterative procedure to get $\tilde{R}_i$. The desired R_i's are

$$R_i = D_c\tilde{R}_iD_c^{-1} \ , \quad i \geq 0 \ .$$

Thus the rate matrix R can be computed up to any level of truncation. In this study the computation of the R_i's was truncated when the iterative procedure stopped in the first iteration, in which case the value was set to 0 .

There is an internal accuracy check which can be used in the above numerical computation. Let

$$\tilde{R}^* = \sum_{i=0}^{\infty} \tilde{R}_1 .$$

From (6) and (7) it follows that

$$\tilde{R}^* = \tilde{L}_1 + \tilde{R}^* \tilde{L}_2 + (\tilde{R}^*)^2 \tilde{M}_1 . \tag{8}$$

The usual iterative method can be used to solve (8) for $\tilde{R}^*$. This matrix can then be compared to the sum of the individual solutions $\tilde{R}_i$, $i \geq 0$. This check was performed and virtually no error detected.

The M/G/1 Paradigm

According to the theory of matrix-geometric invariant vectors, if

$$\underline{0} = \underline{\pi} P_S \tag{9}$$

then

$$\underline{\pi} = (\underline{\pi}_{-1} , \underline{\pi}_0 , \underline{\pi}_1 , \underline{\pi}_2 , \ldots)$$

where $\underline{\pi}_i$ corresponds to the vector on the super block I_i , $i \geq -1$, and

$$\underline{\pi}_{i+1} = \underline{\pi}_i R , \quad i \geq 0 .$$

This can be used with (9) to temporarily reduce the problem of solving (9) to consideration of

$$(\underline{0} \quad \underline{0}) = (\underline{\pi}_{-1} \quad \underline{\pi}_0) \; P_{I_{-1} \cup I_0}$$

where

$$P_{I_{-1} \cup I_0} = \begin{pmatrix} B_{-1,-1} & B_{-1,0} \\ B_{0,-1} & B_{0,0} + RA_2 \end{pmatrix}$$

is a matrix with negative diagonal entries, nonnegative off-diagonal entries, and row sums equal to 0 ; thus it can be thought of as a transition rate matrix for a Markov process on $I_{-1} \cup I_0$. Letting

$$C_0 = M_2 \ , \ C_1 = -D_c + R_0 M \ , \ C_2 = L_2 + R_1 M_1 \ , \ C_i = R_{i-1} M_1 \ , \ i \geq 3 \ ,$$

gives

$$B_{C,0} + RA_2 = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 & \cdots \\ C_0 & C_1 & C_2 & C_3 & \cdots \\ 0 & C_0 & C_1 & C_2 & \cdots \\ 0 & 0 & C_0 & C_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Note that this has the M/G/1 paradigm structure discussed by Lucantoni and Neuts [4]. Following their approach, it will be easier to work with an embedded Markov chain. Define the submatrices of transition probabilities for this chain as

$$\tilde{K}_i = D_i^{-1} K_i \ , \quad \tilde{J}_i = D_i^{-1} J_i \ , \quad 0 \leq i \leq c \ ,$$

$$\tilde{C}_1 = D_c^{-1} C_1 + I \ ,$$

$$\tilde{C}_i = D_c^{-1} C_i \ , \quad i = 0, 2, 3, \ldots$$

This is a Markov chain on

$$I_{-1} \cup I_0 = \bigcup_{m=0}^{c-1} H_m \cup \bigcup_{j=0}^{\infty} H_{0,j} .$$

Each transition of this chain corresponds to a transition of the original process. Each transition of the original process out of a state in $I_{-1} \cup I_0$ is also a transition for this chain. If the process made a transition from I_0 to I_1, then the chain will make a transition from I_0 to I_0, the target state being the first return state in I_0. Thus the chain can make transitions from states of I_0 into themselves.

The following analysis uses ideas from Lucantoni and Neuts [4]. Consider the Markov chain $\tilde{P}_{I_{-1} \cup I_0}$ on this state space. Let G be the $(c+1) \times (c+1)$ matrix of hitting probabilities of states of $H_{0,j}$ starting at $H_{0,j+1}$, $j \geq 0$. *G must satisfy*

$$G = \sum_{\nu=0}^{\infty} \tilde{C}_\nu G^\nu ,$$

and can be found iteratively starting with an initial 0 matrix; see [4]. Let V be the first passage probabilities of hitting states in $H_{0,j+1}$ starting from states in $H_{0,j}$ with $H_{0,j}$ a taboo set; then

$$V = \sum_{\nu=2}^{\infty} \tilde{C}_\nu G^{\nu-2}$$

Let W be the first passage probabilities of hitting states in $H_{0,j+1}$ with $H_{0,j}$ a taboo set starting from states in $H_{0,j+1}$; then

$$W = \sum_{\nu=1}^{\infty} \tilde{C}_\nu G^{\nu-1}$$

Finally, let $\tilde{S}_1$ be the matrix whose entries are the expected number of visits to states in $H_{0,j+1}$ from states in $H_{0,j}$ before returning to $H_{0,j}$; then

$$\tilde{S}_1 = \sum_{\nu=0}^{\infty} VW^{\nu} = V(I-W)^{-1}$$

Using the above relationships $\tilde{S}_1$ can be computed.

Probabilities of Idle Servers

Now consider an invariant vector for the Markov chain with state space $I_{-1} \cup I_0$ and the Markov transition matrix $\tilde{P}_{I_{-1} \cup I_0}$:

$$(\underline{z}_{-1}\ ,\ \underline{z}_0) = (\underline{z}_{-1}\ ,\ \underline{z}_0)\ \tilde{P}_{I_{-1} \cup I_0} \tag{10}$$

Partition the vector over the blocks H_m , $0 \leq m \leq c-1$, and $H_{0,j}$, $j \geq 0$:

$$(\underline{z}_{-1}\ ,\ \underline{z}_0) = (\underline{z}_{-1,0}\ ,\ \underline{z}_{-1,1}\ ,\ \cdots\ ,\ \underline{z}_{-1,c-1}\ ,\ \underline{z}_{0,0}\ ,\ \underline{z}_{0,1}\ ,\ \underline{z}_{0,2}\ ,\ \ldots)$$

From the structure of this process and an important property of taboo probabilities (Theorem 1 of [5], or see Chung [1], p. 53), it follows that

$$\underline{z}_{0,1} = \underline{z}_{0,0}\ \tilde{S}_1$$

This result applied to (10) gives

$$(\underline{z}_{-1},\ \underline{z}_{0,0}) = (\underline{z}_{-1},\ \underline{z}_{0,0}) \begin{pmatrix} 0 & \tilde{K}_0 & & & & & 0 \\ \tilde{J}_1 & 0 & \tilde{K}_1 & & & & \\ & \tilde{J}_2 & 0 & \tilde{K}_2 & & & \\ & & & \cdots & & & \\ & 0 & & \tilde{J}_{c-1} & 0 & & \tilde{K}_{c-1} \\ & & & & \tilde{J}_c & \tilde{C}_1 & + \tilde{S}_1\tilde{C}_0 \end{pmatrix}$$

The dimension of the above square matrix is $(c+1)(c+2)/2$. For a moderate number of servers c, the invariant vector $(\underline{z}_{-1}, \underline{z}_{0,0})$ can be computed using existing numerical techniques. The corresponding invariant vector for the process has component vectors

$$\underline{y}_{-1,i} = \underline{z}_{-1,i} D_i^{-1}, \quad 0 \leq i \leq c-1$$

$$\underline{y}_{0,0} = \underline{z}_{0,0} D_c^{-1} .$$

In order to get a normalizing constant to convert this into the invariant probabilities, recall that the proportion of idle servers must equal

$$1 - \rho = 1 - \left(\frac{\lambda_1}{c\mu_1} + \frac{\lambda_2}{c\mu_2} \right)$$

where ρ is the utilization factor. Thus, let

$$\xi = \sum_{i=0}^{c-1} \frac{c-i}{c} \underline{y}_{-1,i} .$$

Then the invariant probabilities on $I_{-1} \cup H_{0,0}$ can be computed:

$$\underline{\pi}_{-1,i} = \underline{y}_{-1,i} (1-\rho)/\xi , \quad 0 \leq i \leq c-1 ,$$

$$\underline{\pi}_{0,0} = \underline{y}_{0,0} (1-\rho)/\xi .$$

Probabilities of States with Customers Waiting

Now it is possible to build the state space back up, computing the invariant probabilities of additional states. In order to compute the invariant probability vector $(\underline{\pi}_{0,1}, \underline{\pi}_{0,2}, \underline{\pi}_{0,3}, \ldots)$ over

$$\bigcup_{j=1}^{\infty} H_{0,j}$$

it is necessary to depart from the approach of Lucantoni and Neuts [4] because $\tilde{C}_0$ is singular. Instead, consider $\tilde{S}_i$, the matrix whose entries are the expected number of visits to states in $H_{0,j+i}$ from states in $H_{0,j}$ before hitting $H_{0,k}$, $k < j+i$;

$$\tilde{S}_i = \sum_{\nu=i+1}^{\infty} \tilde{C}_\nu G^{\nu-i-1} (I-W)^{-1} .$$

This follows by a simple sample path argument similar to the derivation of $\tilde{S}_1$ earlier. The invariant vector

$$(\underline{z}_{-1} , \underline{z}_{0,0} , \underline{z}_{0,1} , \underline{z}_{0,2} , \ldots)$$

for the Markov chain

$$\tilde{P}_{I_{-1} \cup I_0}$$

must satisfy

$$\underline{z}_{0,i+1} = \sum_{j=0}^{i} \underline{z}_{0,j} \tilde{S}_{i+1-j} , \quad i \geq 0 .$$

This is a special case of the fundamental result for taboo probabilities (Theorem 1 of [5] or see Chung [1], p.53). The invariant probabilities for the process must therefore satisfy

$$\underline{\pi}_{0,i+1} = \sum_{j=0}^{i} \underline{\pi}_{0,j} D_c \tilde{S}_{i+1-j} D_c^{-1} , \quad i \geq 0 .$$

These can be computed recursively, starting from $\underline{\pi}_{0,0}$ which has already been computed.

Finally $\underline{\pi}_{i,j}$, $i \geq 1$, $j \geq 0$, can be computed using the matrix-geometric structure:

$$\underline{\pi}_{i+1,j} = \sum_{k=0}^{j} \underline{\pi}_{i,k} R_{j-k} .$$

Thus one can compute the invariant probability vector to any level of truncation $0 \le i \le I$, $0 \le j \le J$.

Sums and Means

By summing the above probabilities it is possible to get a separate calculation of the probability of no idle servers (this can be used as a consistency check on the numerical calculation) and the mean number of each type of customer awaiting service (then Little's formula can be applied to compute mean delay for each class).

$$\sum_{i,j=0}^{\infty} \underline{\pi}_{i,j} \underline{e}^t = \underline{\Pi}\,(I-R^*)^{-1}\,\underline{e}^t \tag{11}$$

$$\bar{q}_1 = \sum_{i=1}^{\infty} i \sum_{j=0}^{\infty} \underline{\pi}_{i,j} \underline{e}^t = \underline{\Pi}\, R^*(I-R^*)^{-2} \underline{e}^t \tag{12}$$

$$\bar{q}_2 = \sum_{j=1}^{\infty} j \sum_{i=0}^{\infty} \underline{\pi}_{i,j} \underline{e}^t = \lambda_2^{-1}\, \underline{\Pi}\, (S^{(1)}(I-S^{(0)})^{-1} - S^{(0)})\, M_2 \underline{e}^t \tag{13}$$

where $\underline{e}^t$ is a $(c+1)$ dimensional column vector consisting of all 1's , and

$$\underline{\Pi} = \underline{\pi}_{0,0}\,(I-S^{(0)})^{-1}$$

$$S^{(0)} = D_c \sum_{k=1}^{\infty} \tilde{S}_k\, D_c^{-1}$$

$$S^{(1)} = D_c \sum_{k=1}^{\infty} k\, \tilde{S}_k\, D_c^{-1}$$

Computational Experience

The algorithm described above has been programmed in single-precision Fortran and run on GWU's IBM370/3031. Cases with $c=5$ were run. Seventy five different cases were run corresponding to all combinations of $\rho = .2$, $.5$, $.8$, $\lambda_1/\lambda_2 = .25$, $.5$, $1,2,4$, and $\mu_1/\mu_2 = .25$, $.5$, $1,2,4$. Execution times for computing state probabilities and expectations varied from approximately 10 seconds per case with $\rho = .2$ to approximately 60 seconds per case with $\rho = .8$.

Various consistency checks were used in the computational procedure: i) independent calculations of R^* and $\sum_{i=0}^{\infty} R_i$ were compared; ii) the row sums of G were compared with unity; iii) the row sums of $\tilde{J}_c + \tilde{C}_1 + \tilde{S}_1 \tilde{C}_0$ were compared with unity; iv) the total probability computed was compared to unity; and v) moments were computed directly from the state probabilities and compared with values computed from equations (11) and (12). These consistency checks generally agreed to 5 or more digits.

The mean delay for cases with homogeneous service rates were calculated using Cobham's [2] approach. The values agreed with those computed by the above algorithm except in the case $\rho = .8$ where a discrepancy appeared in the fourth digit. This case illustrates the need for a more complete error analysis in this type of calculation.

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Relative Mean Delays

(5 servers, .2 utilization)

		λ_1/λ_2				
		.25	.50	1.0	2.0	4.0
	.25	.0003698 .001947	.0004961 .002613	.0006405 .003333	.00007681 .003922	.0008479 .004441
	.50	.0004947 .001251	.0005728 .001453	.0006720 .001705	.0007704 .001948	.0008510 .002134
μ_1/μ_2	1.0	.0007980 .0009978	.0008211 .001026	.0008514 .001064	.0008841 .001105	.0009123 .001141
	2.0	.001454 .0009104	.001394 .0008738	.001308 .0008214	.001209 .0007597	.001118 .0007021
	4.0	.002846 .0008955	.002665 .0008431	.002384 .0007599	.002020 .0006490	.001650 .0005315

Table 1. Values of $\mu_1 W_1$ and $\mu_2 W_2$ for $c=5$, $\rho = .2$, and $\lambda_1/\lambda_2 = .25$, .5 , 1, 2, 4, and $\mu_1/\mu_2 =$ .25 , .5 , 1, 2, 4.

Relative Mean Delays

(5 servers, .5 utilization)

λ_1/λ_2

		.25	.50	1.0	2.0	4.0
	.25	.01488 .1474	.02195 .2136	.03082 .2854	.03892 .3425	.04414 .3782
	.50	.01844 .07843	.02286 .09859	.02922 .1258	.03636 .1540	.04250 .1765
μ_1/μ_2	1.0	.02898 .05795	.03128 .06257	.03476 .06953	.03911 .07822	.04346 .08690
	2.0	.05220 .05202	.05224 .05192	.05226 .05172	.05226 .05168	.05223 .05165
	4.0	.1011 .05083	.09835 .04964	.09364 .04759	.08669 .04442	.07809 .04030

Table 2. Values of $\mu_1 W_1$ and $\mu_2 W_2$ for $c=5$, $\rho = .5$, $\lambda_1/\lambda_2 = .25, .5, 1, 2, 4$, and $\mu_1/\mu_2 = .25, .5, 1, 2, 4$.

Relative Mean Delays

(5 servers, .8 utilization)

		λ_1/λ_2				
		.25	.50	1.0	2.0	4.0
μ_1/μ_2	.25			.2113 3.9591		
	.50		.1159 1.3541	.1726 1.8911	.2487 2.6118	
	1.0	.1319 .6596	.1511 .7553	.1846 .9218	.2405 1.1956	.3052 1.5113
	2.0		.2462 .5983	.2663 .6369	.2976 .6880	
	4.0			.4589 .5520		

Table 3. Values of $\mu_1 W_1$ and $\mu_2 W_2$ for $c=5$, $\rho = .5$, $\lambda_1/\lambda_2 = .25$, .5, 1, 2, 4, and $\mu_1/\mu_2 = .25$, .5, 1, 2, 4.

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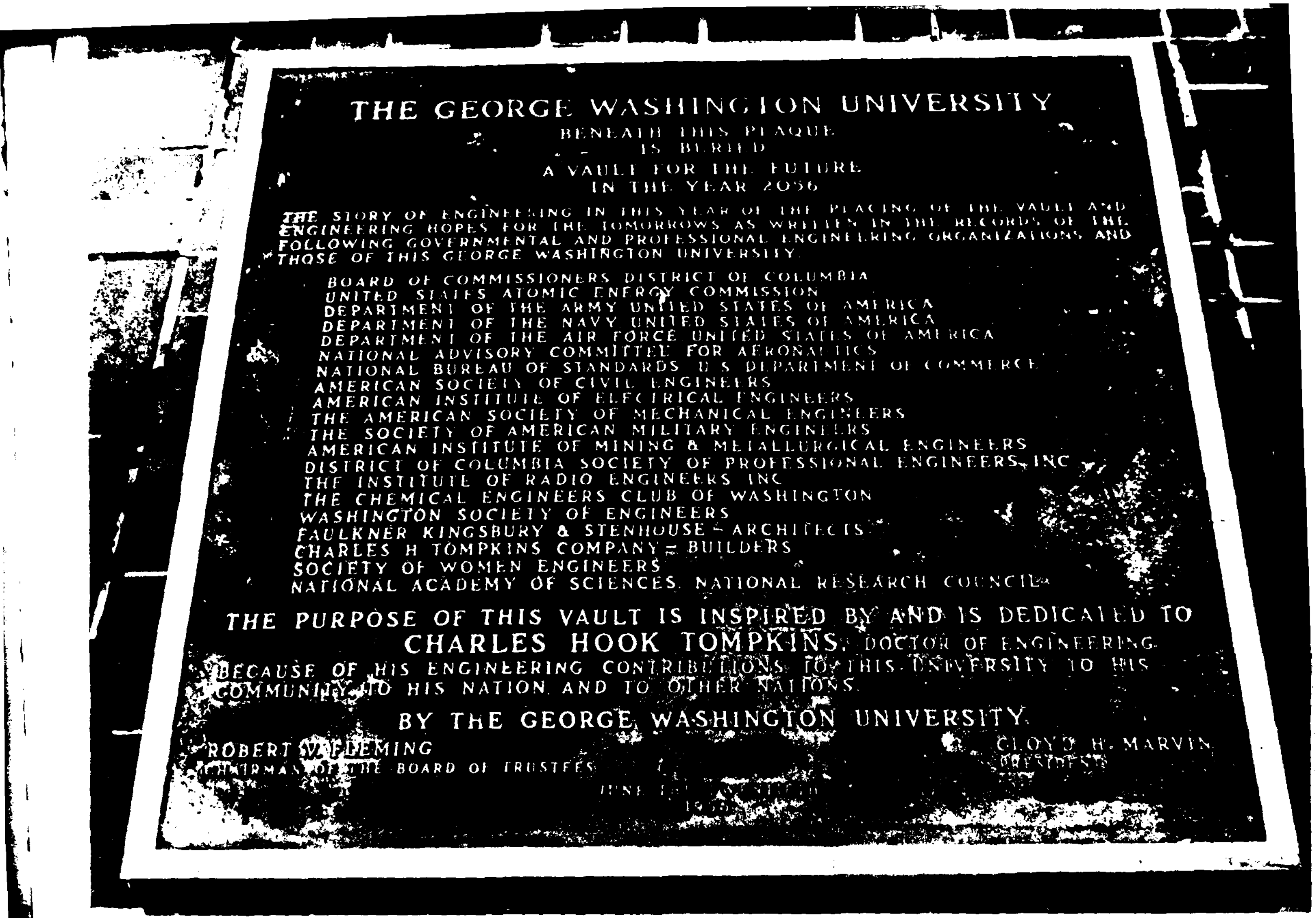

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